Measuring the Robustness of Neural Networks via Minimal Adversarial Examples

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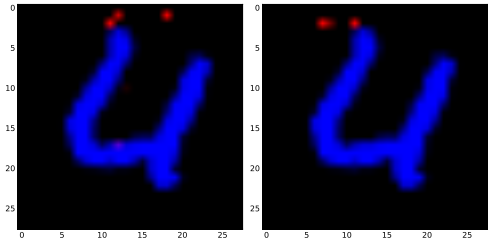
Abstract

Neural networks are highly sensitive to adversarial examples, which cause large output deviations with only small input perturbations. However, little is known *quantitatively* about the distribution and prevalence of such adversarial examples. To address this issue, we propose a rigorous search method that provably finds the *smallest* possible adversarial example. The key benefit of our method is that it gives precise quantitative insight into the distribution of adversarial examples, and guarantees the absence of adversarial examples if they are not found. The primary idea is to consider the nonlinearity exhibited by the network in a small region of the input space, and search exhaustively for adversarial examples in that region. We show that the frequency of adversarial examples and robustness of neural networks is up to twice as large as reported in previous works that use empirical adversarial attacks. In addition, we provide an approach to approximate the nonlinear behavior of neural networks, that makes our search method computationally feasible.

Neural networks are not robust: Szegedy et al. [2013] first showed that neural networks can be made to incorrectly classify correctly labeled inputs by perturbing the input by a small amount. Hence, there are no safety and security guarantees when applying neural networks in real-world applications (e.g. autonomous vehicles). A key issue is to compute objective measures for network robustness, e.g. how many adversarial examples arise (*frequency)* and the size of (the smallest) adversarial example (*severity*). However, this is challenging, as state-of-the-art networks can be large and highly nonlinear.

To address this issue, we propose a rigorous search method to construct adversarial examples for a class of neural networks: fully-connected networks with ReLU activations. Our method combines three insights: 1) ReLU neural networks can be viewed as a collection of logical clauses (as proposed in Bastani et al. [2016]). 2) By considering the behavior of network activations in regions around an input, finding an adversarial example can be reduced to solving a large system of independent linear programs. 3) We propose an approximation to the nonlinear behavior of network activations, as constructing the set of feasible piecewise linear regions for the hierarchy of network layers can be prohibitively expensive. Combining these, we obtain a tractable approach to construct adversarial examples.



(a) Baseline $|\Delta x| = 3.6$ (b) Ours $|\Delta x| = 2.4$

Figure 2: Adversarial examples that switch prediction from 4 to 6. Our method consistently finds smaller adversarial perturbations than baselines. Blue: original; red: Δx.

The key benefits of our approach are two-fold: 1) our method provably finds the smallest adversarial example for any specified output change (or guarantees the absence of such adversarial examples) and hence 2) it gives unbiased quantitative measures of the robustness of the neural network.

Several adversarial attacks have previously been proposed: Goodfellow et al. [2014], Carlini and Wagner [2016] use input space gradients, while Bastani et al. [2016] use a constraint programming based approach, similar to us. However, these methods do not provide theoretical guarantees on completeness nor optimality. [Katz et al., 2017, Ehlers, 2017, Huang et al., 2017] use formal methods to analyze the network in its entirety, but these do not scale well for larger networks. Instead, our method both guarantees minimality of the adversarial example and scales better to large networks.

In this work, we present a tractable approach to find adversarial examples and analyze local network robustness via a model reduction to constraints in linear arithmetic. Our contributions are as follows:

- We propose an adversarial attack that uses a reduction of neural networks into logical clauses and an iterative search over *feasible* network activations to find adversarial examples.
- We make this search tractable via approximation bounds on the set of feasible activations.
- We show that our method provably finds the minimal adversarial example for an input x or guarantees its absence within its neighborhood.
- We validate our method by showing that ReLU neural networks can be attacked with examples with a smaller magnitude, and up to twice as frequently as previously reported.

1 Finding Adversarial Examples via Satisfiability Problems

Our goal is to find the minimal adversarial perturbation $\Delta x \in \mathbb{R}^d$ for a given neural network classifier $f(x)$ that changes f's output to a target class j:

$$\Delta x^* = \underset{\Delta x}{\operatorname{argmin}} \{\|\Delta x\|_p \mid f(x) = i, f(x + \Delta x) = j, i \neq j\}. \tag{1}$$

Bastani et al. [2016] quantified the distribution of such examples using the point-wise robustness $\rho(x, f, j)$, the smallest ball around x that does not contain adversarial examples:

$$\rho(x, f, j) = \inf\{\epsilon \mid \exists \Delta x : f(x + \Delta x) = j, \|\Delta x\|_p \leq \epsilon\}. \tag{2}$$

where $p = 1, 2, \infty$ has been considered in literature. In our work we use $p = 1$, but the principles hold for the other norms as well. Given a distribution D from which data points are sampled, based on ρ, the adversarial frequency ϕ and severity μ are the prevalence and average robustness:

$$\phi(f, \delta, j) = \mathbb{P}_{x \sim D}(\rho(x, f, j) \leq \delta), \quad \mu(f, \delta, j) = \mathbb{E}_{x \sim D}(\rho(x, f, j) | \rho(x, f, j) \leq \delta) \tag{3}$$

ReLU networks as logic constraints Hereafter, we will consider fully-connected networks with ReLU activations. We follow Bastani et al. [2016] and describe a ReLU network as a logic formula in linear arithmetic. Here, each ReLU node y corresponds to a logical disjunction:

$$y = \max(0, w^T x) \Leftrightarrow (y = 0 \wedge w^T x \leq 0) \vee (y = w^T x \wedge y > 0), \tag{4}$$

where each clause represents a linear region (polytope) of its input space. A change in the input x can cause a *switch in activation*: the active clause can change from $w^T x > 0$ to $w^T x \leq 0$, or vice versa. Similarly, a network with m ReLU nodes across all layers can be represented as a single clause with m disjunctions, whose 2^m combinations of node activations each represent a linear region for the network. Finding an adversarial example then corresponds to finding an x' that satisfies this m-node clause and activates a desired target node in the output layer.

An iterative approach would 1) sweep through the network sequentially node-by-node, 2) check what combination of node activation states can be attained for the network and 3) search over these combinations for the minimal adversarial example. However, this is computationally expensive and the sequential nature of the search prohibits parallelization. A naive parallelized search would consider 2^m combinations of activation patterns and is prohibitively expensive.

Algorithm 1 Minimal Adversarial Example Search

Input: Neural Network f with L layers, perturbation radius ϵ, target-class j.

for layer $l \in L$ layers **do**
    `GetInputRegions`: Construct constraints $C_l(x_l)$ using Lemma 2 that bound the inputs $\{x_l\}$.
    `GetSwitchingNodes`: Collect nodes in layer l that switch activation (equation 4).
end for
for layer $l \in L$ layers **do**
    `CheckAct`: Given $C_l(x_l)$, check feasible activation states for switching nodes in layer l.
    `GetFeasibleRegions`: Get constraints $\tilde{S}_l(x_l)$ for *feasible* linear-regions under $C_l(x_l)$.
end for
Construct combinations $(P_1, P_2, \ldots, P_t)$ of $\left\{\tilde{S}_l\right\}$.
Search $\{P_i\}$ for minimal adversarial input perturbation Δx that changes output to target class j.

Searching for provably minimal adversarial examples We now propose a tractable search algorithm (outlined in Algorithm 1) to find minimal adversarial examples in a candidate set of perturbations $B(x, \epsilon) = \{x' | \|x' - x\| \leq \epsilon\}$ around an input x. The main idea is to use a parallelized approach and limit the number of activations to search over. Given a set of inputs $C_l(x_l)$ for layer l, we compute a convex-relaxation $C_{l+1}(x_{l+1})$ of the image of $C_l(x_l)$ through the layer $\{\text{ReLU}(x_l)\}$. These relaxations are then used to parallelize.

More specifically, our approach has the following key steps. First, we find the nodes for each layer that can switch activation state (`GetSwitchingNodes`), on a layer-by-layer basis. For the first ReLU layer $y = \max(0, Wx + u)$, we define the constraint $C(x) := \|\Delta x\| \leq \epsilon$. To check if a node y_i switches, we check whether both the constraints $W_i^T x + u_i > 0 \wedge C(x)$ and $W_i^T x + u_i \leq 0 \wedge C(x)$ have satisfying assignments. If both are satisfiable, then the node can potentially switch activation for some perturbation.

Then, for the next layer, the constraint $C(y)$(`GetInputRegions`) is determined by bounding Δy using a relaxation $\|\Delta y - A\Delta x\| \leq \|\Delta \bar{A}\|\epsilon$ (Lemma 2) and $\|\Delta x\| \leq \epsilon$. We then use $C(y)$ to find the switching nodes, and repeat this layer-wise. Next, in a layer-wise manner, we compute *feasible* combinations s^l_{switch} of node activations for the nodes that can switch in each layer (`CheckAct`). Such combinations determine a linear region $\tilde{S}_l$ for the layer output, which can be represented as $s^l_{\text{switch}} \wedge y = Ax + b$ (`GetFeasibleRegions`).

Finally, we construct linear regions P_i by combining regions defined in $\{\tilde{S}_l\}$ across layers and search for the smallest adversarial example δ_i within P_i. Since we have searched over all the linear-regions of the network that can be activated by an input in $B(x, \epsilon)$, we have:

Lemma 1. *If $\Delta x^* = \min\{\Delta x_i : i = 1, 2, \ldots, t\} \leq \epsilon$, then Δx_i is the globally minimal adversarial example for the targeted attack. Otherwise, there exists no adversarial examples in $B(x, \epsilon)$.*

Parallelization This approach can be parallelized at two levels: 1) Given a set of inputs for a layer l, the set of switching nodes for a layer can be determined in parallel across nodes. 2) After we have computed the over-approximation of the image of $B(x, \epsilon)$ at each layer, we can compute the possible activation patterns for each of the layers in parallel.

Approximating perturbations In order to improve search efficiency, in Algorithm 1 we consider a relaxation of the network's nonlinear behavior over the input region $B(x, \epsilon)$. To do so, we derive an upper bound on the output perturbation Δy given a set of perturbed inputs $x' = x + \Delta x$ for a single ReLU layer $y = \max(0, Wx + u)$:

Lemma 2. *Consider a ReLU layer $y = \max(0, Wx + u)$ whose local linear behavior at a given x can be represented as $y = Ax + b$. For any input perturbation $\|\Delta x\| \leq \epsilon$, Δy satisfies*

$$\|\Delta y - A\Delta x\| \leq \epsilon \|\Delta \bar{A}\|. \tag{5}$$

where the rows of $\Delta \bar{A}$ are the weights for the nodes that can switch for a perturbation $\|\Delta x\| \leq \epsilon$.

Proof. Given $x' = x + \Delta x$, the ReLU output changes as:

$$y + \Delta y = \max(0, W(x + \Delta x) + u) = (A + \Delta A)x + (b + \Delta b), \tag{6}$$

Method ($\delta = 5$)	Frequency ρ
Baseline	26.85%
Our approach	**36.55%**

(a) Frequency ρ in %.

Metric ($\delta = 20$)	Method / Target class	5	6
Severity μ	Baseline	**12.16**	17.52
Severity μ	Ours	**8.35**	13.82

(b) Severity μ in net pixel change.

Target class	0	1	2	3	4	5	6	7	8	9
Baseline ($\delta = 5$)	1.8	3.4	6.0	13.4	2.9	17.6	2.4	9.5	7.2	11.5
Our approach ($\delta = 5$)	**2.2**	**3.9**	**9.6**	**18.0**	**5.1**	**23.8**	**6.6**	**14.8**	**11.9**	**15.7**
Baseline ($\delta = 10$)	7.9	7.7	8.6	47.0	10.1	42.5	11.5	22.6	38.2	23.9
Our approach ($\delta = 10$)	**16.8**	**11.1**	**60.0**	**63.2**	**18.5**	**62.8**	**27.3**	**40.0**	**56.4**	**48.8**

(c) Frequency ρ in %.

Figure 3: a) Frequency for untargeted attacks. b) Severity and robustness for targeted attacks. c) Frequency for targeted attacks.

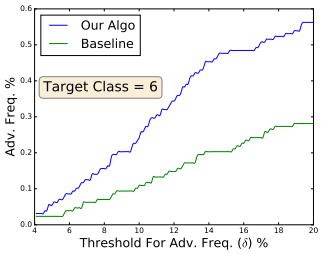


Figure 5: Adversarial frequency for various thresholds δ.

where $\Delta A, \Delta b$ capture the nonlinear behavior of the activation function. Consider a node y_i that can be made to switch with some such perturbation Δx. There exists some $0 \leq \alpha_i \leq 1$, such that

$$(A_i + \Delta A_i)(x + \alpha_i \Delta x) + (b_i + \Delta b_i) = 0 = A_i(x + \alpha_i \Delta x) + b_i, \tag{7}$$

Combining the relations above, we get

$$\Delta y_i = (A + (1 - \alpha_i)\Delta A_i)\Delta x_i = (A + \Psi_i \Delta \bar{A}_i)\Delta x_i. \tag{8}$$

For non-switching nodes, we have $\Delta y_i = A_i \Delta x + \Delta b_i$. Note that for the ReLU activation $\Delta A_i = \pm \boldsymbol{W}_i$. Hence, for any given perturbation Δx, we have $(1 - \alpha_i)\Delta A_i \Delta x_i = \Psi_i \Delta \bar{A}_i \Delta x_i$, with $\Psi_i = \pm(1 - \alpha_i)$ for switching nodes and $\Psi_i = 1 - \alpha_i = 0$ for nodes that do not switch. Equivalently, we can write $\Delta y - A\Delta x = \Psi \Delta \bar{A} \Delta x$, where Ψ is a diagonal matrix with Ψ_i as its ith diagonal entry. Since $\|\Psi\|_p \leq 1$ for $p = 1, 2, \infty$, equation (8) reduces to $\|\Delta y - A\Delta x\| \leq \|\Delta \bar{A}\| \cdot \|\Delta x\| \leq \|\Delta \bar{A}\| \epsilon$. □

We also enforce $y + \Delta y \geq 0$. Here, we considered a convex-relaxation of the set of feasible perturbations ΔA. Note that Lemma 2 can be applied iteratively across layers and is tighter than the bound used in Szegedy et al. [2013], which used the operator norm of W i.e $\|\Delta y\| \leq \|W\| \cdot \|\Delta x\|$.

2 Experiments

We trained a 3-layer network with 60-30-10 hidden units to an accuracy of 97% on MNIST [LeCun and Cortes, 2010]. For 300 randomly sampled examples, we compute the adversarial frequency for targeted and untargeted attacks, and severity and robustness for targeted attacks. We use the l_1-norm to measure distances: changing one pixel from black to white corresponds to an l_1-norm of 1. In our experiments, we use the l_1-ball $B(x, \epsilon = 5.0)$ around each sample x to find the minimal adversarial example and compare all results to Bastani et al. [2016], which is a recent state-of-the-art approach to measuring robustness.

We find that on all metrics, our method outperforms the baseline. First, our method achieves significantly higher frequencies for both targeted (Table 3c) and untargeted (Table 3a) attacks. Here, our method includes linear-regions feasible with $\epsilon = 5$, but we compare frequencies against baseline for examples with threshold $\Delta x \leq \delta = 5, 10$. For the $\epsilon < \delta = 10.0$ case, our method still succeeds more frequently even though we consider fewer candidate perturbations than strictly allowed for. For this case, the guarantees regarding unbiasedness of the measurements do not hold.

This trend is consistent as δ changes: Figure 5 shows our method achieves higher adversarial frequency for all δ, for two representative target classes; we see a similar trend in Tables 3b and 3c. Moreover, our method finds smaller adversarial examples: the severity (see Table 3b) is significantly smaller. Hence, network is significantly less robust than estimated by baselines (see Table 3b). Note that, for fair comparison, we compute severity only with those examples where the baseline attack succeeds with $\delta < 20.0$.

Figure 2 shows a representative instance where our attack finds a smaller adversarial perturbation than the baseline. We see that the perturbation found is different in nature from the baseline.

3 Future work

The constraints corresponding to a linear region and the output being assigned a specific label represent a decision boundary for the network. This could allow us to compute population statistics,

such as the local density of adversarial samples. To scale to large networks, efficient algorithms to construct *interpolants* in linear-arithmetic[Albarghouthi and McMillan, 2013, Lynch and Tang, 2008] can be used to approximate the boundaries. Moreover, our approach could aid the design of robust networks and analysis of defenses against adversarial attack.

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